BERNSTEIN-CHEBYSHEV INTEGRAL COLLOCATION METHOD FOR SOLVING THIRD-ORDER MULTI-POINT BOUNDARY VALUE PROBLEMS

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Abstract: In this paper, combination of Bernstein and Chebyshev approximations are developed and adopted for the numerical solution of third-order linear and nonlinear multi-point boundary value problems. The whole idea of the method is based on the Bernstein-Chebyshev approximation for the third-order derivatives and we generate approximations to the second-order, first-order and function y itself through successive integration of third-order derivative. Newton's linearization scheme is employed to linearize the nonlinear equations and then resulting to iterative procedure. Numerical examples of linear and nonlinear problems are considered to illustrate the efficiency and reliability of the method and the results obtained are compared with other methods in the literature.

Keywords: Bernstein-Chebyshev approximation, Newton's linearization scheme, linear, nonlinear problems

Introduction

Multi-point boundary value problems have received lots of attention and they are still receiving such an attention due to the fact that many physical problems arising from electric power networks, railway systems, telecommunication lines, construction of large bridges with many supports, theory of elastic stability occurring in a wire of cross section etc can be formulated into mathematical equations result in this type of boundary value problem. The numerical solvability of multipoint boundary value problems has been considered by several authors and different methods have been developed and applied by these authors. Among these methods are the Reproducing Kernel Method (Akram *et al.*, 2013; Li *et al.*, 2012; Li and Wu, 2012; Wu and Li, 2010), Weighted Residuals (Haque *et al.*, 1986), Pade approximation (Trimizi *et al.*, 2005), Adomain Decomposition Method (Tatari and Dehghan, 2006), Homotopy Perturbation Method (Siddiqi and Iftikhar, 2014), Recently, Ogunlaran and Oladejo (2014) developed a method using Chebyshev polynomials for solving both linear and nonlinear boundary value problems and several other methods have been proposed for handling third order boundary value problems.

The main aim of this paper is to develop a new method for solving third-order multi-point boundary value problems and this is based on the combination of Bernstein and Chebyshev polynomials. Thus, we consider multi-point boundary value problem of the form:

$$a_0(x)y'''(x)+a_1(x)y''(x)+a_2(x)y'(x)+a_3(x)y(x)=f(x) \qquad (1)$$

subject to the boundary conditions:

$$y(0)=a_1 \qquad (2)$$

$$y'(0)=a_2 \qquad (3)$$

$$y'(1)=a_3y'(\eta)+\lambda, \qquad (4)$$

Where: $a_0(x), a_1(x), a_2(x), a_3(x)$ and $f(x)$ are continuous functions and $\alpha_1, \alpha_2, \alpha_3$ are constants and $\eta \in (0,1)$.

This paper is organised as follows: In section 2, we define Bernstein and Chebyshev polynomials, section 3 presents the construction of Bernstein-Chebyshev Integral Collocation Method and some numerical examples are given to establish the accuracy and efficiency of the proposed method in section 4. Finally in section 5, the concluding remarks are given.

Polynomial Approximation

Polynomials are of paramount importance when it comes to approximation theory. According to Burden and Faires (1993), polynomials are simple to differentiate, integrate, evaluate at arbitrary values and can be used to approximate any continuous function on a closed interval to within arbitrary tolerance. Consequent upon these, they are widely used in modeling of scientific and engineering problems.

Bernstein Polynomials

The general form of Bernstein Polynomials of the *nth* degree on the interval [*a, b*] is defined as

$$B_{i,n}(x)=\binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n}, i=0,1,...,. \qquad (5)$$

These ($n+1$) Bernstein polynomials or **B**-polynomials form a complete bases over the interval [*a*,*b*] ,see ([5]). The **B**-polynomials are generated by a recursive relation

$$B_{i,n}(x)=\frac{b-x}{b-a}B_{i,n-1}(x)+\frac{x}{b-a}B_{i-1,n-1}(x) \qquad (6)$$

Chebyshev Polynomials

The Chebyshev Polynomials of the first kind are defined by:

$T_n(x)=\cos(n\cos^{-1}x), n=0,1,2,\cdots,N \qquad (7)$

and they can be determined with the aid of the following recurrence formula:
$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n = 1,2,\cdots$ (8)
which together with the initial conditions
$T_0(x) = 1, T_1(x) = x$ (9)
recursively generates the polynomials $\{T_n(x)\}$ very efficiently. These could be converted into any interval of consideration. For example, in $[a,b]$, we have

$$T_n(x) = \cos\left[n\cos^{-1}\left(\frac{2x-a-b}{b-a}\right)\right] \quad (10)$$

and the recursive relation is given as

$$T_{n+1}(x) = 2\left\{\frac{2x-a-b}{b-a}\right\}T_n(x) - T_{n-1}(x) \quad (11)$$

Alternatively, Chebyshev polynomials are defined in terms of Bernstein polynomials as

$$T_n(2x-1) = \sum_{i=0}^{n}(-1)^{n+i}\frac{\binom{2n}{2i}}{\binom{n}{i}}B_{i,n}(x), \quad (12)$$

with $x \in [0,\ 1]$. See (Lin *et al*. 2012a).

Construction of Bernstein-Chebyshev Integral Collocation Method

Following May-Duy (2004), the construction process of the combined Bernstein-Chebyshev Integral Collocation Method for third-order multi-point boundary value problems is discussed as follows:

$$\frac{d^3y(x)}{dx^3} = \sum_{i=0}^{n}c_i\left(B_{i,n}(x)+T_i(x)\right) \quad (13)$$

Successive integration of equation (13) gives

$$\frac{d^2y}{dx^2} = \sum_{i=0}^{n}c_i\int\left(B_{i,n}(x)+T_i(x)\right)dx + k_1 = \sum_{i=0}^{n+1}\varsigma_{i,2}\phi_i^{[2]}(x). \quad (14)$$

$$\frac{dy}{dx} = \sum_{i=0}^{n}c_i\int\phi_i^{[2]}(x)dx + k_1x + k_2 = \sum_{i=0}^{n+2}\varsigma_{i,1}\phi_i^{[1]}(x). \quad (15)$$

$$y(x) = \sum_{i=0}^{n}c_i\int\phi_i^{[1]}(x)dx + \tfrac{1}{2}k_1x^2 + k_2x + k_3 = \sum_{i=0}^{n+3}\varsigma_{i,0}\phi_i^{[0]}(x). \quad (16)$$

Substituting equations (13) - (16) into equation (1), we have

$$a_0\left[\sum_{i=0}^{n}c_i\left(B_{i,n}(x)+T_i(x)\right)\right] + a_1\left[\sum_{i=0}^{n+1}\varsigma_{i,2}\phi_i^{[2]}(x)\right] + a_2\left[\sum_{i=0}^{n+2}\varsigma_{i,1}\phi_i^{[1]}(x)\right]$$
$$+ a_3\left[\sum_{i=0}^{n+3}\varsigma_{i,0}\phi_i^{[0]}(x)\right] = f(x). \quad (17)$$

Thus, collocating equation (17) at points $x = x_j$, we obtain

$$a_0\left[\sum_{i=0}^{n}c_i\left(B_{i,n}(x_j)+T_i(x_j)\right)\right] + a_1\left[\sum_{i=0}^{n+1}\varsigma_{i,2}\phi_i^{[2]}(x_j)\right] + a_2\left[\sum_{i=0}^{n+2}\varsigma_{i,1}\phi_i^{[1]}(x_j)\right]$$
$$+ a_3\left[\sum_{i=0}^{n+3}\varsigma_{i,0}\phi_i^{[0]}(x_j)\right] = f(x_j), \quad (18)$$

where

$$x_j = a + \frac{(b-a)j}{n+2},\ j = 1,2,\cdots n+1. \quad (19)$$

Thus, equation (17) constitutes a system of (n+1) equations in (n+1) unknown constants. Extra 3 equations are obtained from the boundary conditions. Altogether, we have a system of (n+4) equations which can be solved by Guassian elimination method to obtain the (n+4) unknown constants which are then substituted into equation (16) to obtain the approximate solution.

Numerical Results and Discussion

The efficiency of the numerical method constructed in section 3 is investigated using linear and nonlinear multi-point boundary value problems.

Problem 4.1: Consider the following linear third-order boundary vsalue problem (Abdullah *et al.*, 2013; Abd El-Salam, 2010; Akram *et al.*, 2013; Ogunlaran & Oladejo, 2014)

$$y'''(x) - xy(x) = (x^3 - 2x^2 - 5x - 3)e^x \qquad (20)$$

subject to the boundary conditions:

$$y(0) = 0, \quad y'(0) = 1, \qquad y'(1) = -e \qquad (21)$$

The exact solution is $y(x) = x(1-x)e^x$. Now solving equation (20) together with the boundary conditions (21) by using our method with n = 6 and n = 7, we obtain the following approximate solutions, respectively:

$$y_6(x) = x + 0.000001146312x^2 - 0.5000109668x^3 - 0.333276856x^4 - 0.1251796195x^5$$
$$- \ 0.0329692497x^6 - 0.007413418x^7 - 0.008230984x^8 - 0.00032794414x^9, \ (22)$$

and

$$y_7(x) = x - 0.00000006332x^2 - 0.4999993394x^3 - 0.333337346x^4$$
$$-0.124984560x^5 - 0.033372444x^6 - 0.0068784763x^7 - 0.001263431x^8$$
$$-0.0001233397x^9 - 0.00004099335x^{10} \qquad (23)$$

Table 1 shows the comparison among the present method, Chebyshev Collocation Method by Ogunlaran and Oladejo (2014) and Reproducing Kernel Method by Akram *et al.* (2013). It is clear from the table that the results obtained by the present method are superior to those obtained by Chebyshev Collocation and Reproducing Kernel Methods.

Table 1: Comparison of Absolute Errors for Problem 4.1

x	Akram *et al.* (2013)		Ogunlaran and Oladejo (2014)		Our method	
	n=20	n=80	Evenly-spaced nodes (n=8)	Unevenly-spaced nodes (n=8)	n=6	n=7
0	0	0	2.8834E-18	5.8188E-18	0	0
0.1	8.29E-07	9.68E-10	9.9859E-08	5.8083E-09	4.67E-09	2.40E-10
0.2	-	-	2.3056E-07	1.4700E-08	9.10E-09	5.00E-10
0.3	1.63E-07	8.67E-09	3.5605E-07	3.1422E-09	1.20E-08	9.00e-10
0.4	4.88E-07	8.85E-09	4.8630E-07	2.3245E-09	1.36E-08	1.10E-09
0.5	4.62E-07	2.52E-09	6.1654E-07	1.5719E-08	1.38E-08	1.40E-09
0.6	-	-	7.4853E-07	3.3380E-08	1.28E-08	1.20E-09
0.7	8.12E-07	3.57E-07	8.8360E-07	2.6617E-08	1.00E-08	1.00E-09
0.8	-	-	1.0187E-06	1.4216E-08	5.90E-09	2.00E-10
0.9	6.60E-07	7.56E-09	1.1639E-06	2.4924E-08	4.85E-09	2.50E-09
1.0	0	0	1.2782E-06	3.1640E-08	6.24E-09	6.95E-09

Problem 4.2: Consider the following variable coefficient non-homogeneous linear third-order boundary value problem (Akram *et al.*, 2013; Ogunlaran & Oladejo, 2014).

$$y'''(x) + xy''(x) = -6x^2 + 3x - 6, \ 0 \le x \le 1 \qquad (24)$$

subject to the boundary conditions:

$$y(0) = 0, \quad y'(0) = 1, \qquad y'(1) = y'\left(\tfrac{1}{2}\right) - \tfrac{3}{4} \qquad (25)$$

By applying the method in section 3 for case n = 3, we obtain our approximate solution as $y(x) = x + \frac{3}{2}x^2 - x^3$ which is the exact solution to this problem.

Problem 4.3: Consider the following non-homogeneous nonlinear third-order boundary value problem (Ogunlaran & Oladejo, 2014).

$$y'''(x) + \left(y''(x)\right)^2 = \sin^2 x - \cos x, \ 0 \le x \le 1 \qquad (26)$$

subject to the boundary conditions:

$$y(0) = 0, \quad y'(0) = 1, \qquad y'(1) = y'\left(\tfrac{1}{2}\right) + \cos(1) - \cos\left(\tfrac{1}{2}\right). \qquad (27)$$

The exact solution is $y(x) = \sin \ x.$

The nonlinear differential equation (26) is converted into a sequence of linear differential equations generated by Newton's linearization technique as

$$y'''_{k+1}(x)+2y''_k(x)y''_{k+1}(x)=\left(y''_k\right)^2+\frac{1-\cos 2x}{2}-\cos x,\ k=0,1,.... \tag{28}$$

subject to the boundary conditions:

$$y_{k+1}(0)=0,\ y'_{k+1}(0)=1,\ y'_{k+1}(1)=y'_{k+1}\left(\tfrac{1}{2}\right)+\cos(1)-\cos\left(\tfrac{1}{2}\right). \tag{29}$$

Here, we choose our initial approximation as $y_0(x)=x+\left(\cos(1)-\cos\left(\tfrac{1}{2}\right)\right)x^2$.

The numerical results for this problem at third iteration (k=2) are presented in Table 2 for case n = 6 and the results are compared with Ogunlaran and Oladejo (2014). It is clearly observed from the table that the performance of the new method in terms of accuracy is better than Ogunlaran and Oladejo (2014).

Table 2: Comparison of Absolute Errors for Problem 4.3 at third iteration

X	Ogunlaran and Oladejo (2014)		Our method
	Evenly-spaced nodes (n=8)	Unevenly-spaced nodes (n=8)	n=6
0	3.8937E-18	2.2549E-17	0
0.1	8.8540E-10	5.9524E-10	3.00E-11
0.2	2.1237E-09	1.6797E-10	1.00E-10
0.3	2.4167E-09	1.2344E-10	2.00E-10
0.4	4.8387E-09	1.3939E-10	1.00E-10
0.5	6.3711E-09	3.6888E-10	3.00E-10
0.6	8.0310E-09	6.4213E-10	4.00E-10
0.7	9.8410E-09	7.7079E-10	4.00E-10
0.8	1.1793E-08	8.7081E-10	6.00E-10
0.9	1.3972E-08	1.1518E-09	3.00E-10
1.0	1.6161E-08	1.4655E-09	7.00E-10

Conclusion

In this paper for the first time in the literature, Combined Bernstein and Chebyshev polynomials are used to develop a new method for approximate solution of third-order linear and nonlinear multi-point boundary value problems. The comparison of our method as depicted by Tables 1-3 showed that the method produces accurate solution for small value of n and these results also revealed that the proposed method is a powerful mathematical tool for obtaining the solutions of linear and nonlinear multi-point boundary value problems.

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